Position of pole of orbit in celestial sphere .	Long.	352·8
	Lat.	+ 64·6
	R.A.	316·1
	Decl.	+ 53·8

These elements give for the Mass of *Mars* . $\frac{1}{3,090,000}$

The Inner Satellite.

Major semi-axis of apparent orbit at distance [9·5930] .	33″·0 ± 1″
Period of revolution	$7^h\ 38^m{\cdot}5 \pm 0^m{\cdot}5$
Hourly motion in Areocentric longitude	47°·11

Passage through the eastern apsis ($p = 70°$) Aug. 20, $13^h{\cdot}0$, W.M.T.

Very respectfully, your obedient servant,

John Rodgers,

Rear Admiral and Superintendent.

The Hon. R. W. Thompson, Secretary of the Navy.

On the Solution of Kepler's Problem.

By J. W. L. Glaisher, M.A., F.R.S.

§ 1. Writing Kepler's equation in the form

$$x = a + e \sin x, \quad \ldots \ldots \ldots \quad (1)$$

the solution in powers of e is

$$x = a + e \sin a + \frac{e^2}{1.2}\frac{d}{da}(\sin^2 a) + \frac{e^3}{1.2.3}\frac{d^2}{da^2}(\sin^3 a) + \&c.,$$

and, on performing the differentiations, it will be found that

$$\begin{aligned}
x = \ & a \\
& + e.\sin a \\
& + e^2.\sin a \cos a \\
& + e^3.\sin a \cos^2 a - \frac{1}{2}\sin^3 a \\
& + e^4.\sin a \cos^3 a - \frac{5}{3}\sin^3 a \cos a \\
& + e^5.\sin a \cos^4 a - \frac{11}{3}\sin^3 a \cos^2 a + \frac{13}{24}\sin^5 a \\
& + e^6.\sin a \cos^5 a - \frac{20}{3}\sin^3 a \cos^3 a + \frac{47}{15}\sin^5 a \cos a \\
& + e^7.\sin a \cos^6 a - \frac{65}{6}\sin^3 a \cos^4 a + \frac{1291}{120}\sin^5 a \cos^2 a - \frac{541}{720}\sin^7 a \\
& + e^8.\sin a \cos^7 a - \frac{49}{3}\sin^3 a \cos^5 a + \frac{427}{15}\sin^5 a \cos^3 a - \frac{1957}{315}\sin^7 a \cos a \\
& + \&c. \quad \ldots \ldots \ldots \quad (2)
\end{aligned}$$

Thus, as is well known, the expression

$$a + \frac{e \sin a}{1 - e \cos a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

differs from the root of the equation (1) only by quantities of the order e^3. It is also to be noticed that the second term of (3) represents the sum of the terms contained in the first column of (2), continued to infinity.

It is evident, therefore, that if the value of the series (2) had to be calculated up to the terms of the fourth or a higher order, it would be preferable to calculate the value of the second term in (3) rather than the values of the terms in the first column of (2), so that, in the calculation of the terms in the other columns, we may assume that the value of log $(1 - e \cos a)$ has been already obtained; and it is of some interest to examine whether these terms can be arranged in such a manner that the knowledge of the value of log $(1 - e \cos a)$ may be applied to reduce the number of logarithmic entries required for their calculation.

§ 2. It is clear that the terms in e^3 and e^4 in the second column of (2) can be represented by

$$-\frac{1}{2} \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}},$$

so that

$$a + \frac{e \sin a}{1 - e \cos a} - \frac{1}{2} \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

differs from the root of (1) only by quantities of the order e^5: in fact we find that

$$\begin{aligned} x = a &+ \frac{e \sin a}{1 - e \cos a} - \frac{1}{2} \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}} \\ &+ e^5 . - \frac{1}{18} \sin^3 a \cos^2 a + \frac{13}{24} \sin^5 a \\ &+ e^6 . - \frac{20}{81} \sin^3 a \cos^3 a + \frac{47}{15} \sin^5 a \cos a \\ &+ e^7 . - \frac{325}{162} \sin^3 a \cos^4 a + \frac{1291}{120} \sin^5 a \cos^2 a - \frac{541}{720} \sin^7 a \\ &+ e^8 . - \frac{1039}{729} \sin^3 a \cos^5 a + \frac{427}{15} \sin^5 a \cos^3 a - \frac{1957}{315} \sin^7 a \cos a \\ &+ \&c. \quad . \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

§ 3. The terms involving e^5 and e^6 in (5)

$$= e^5 . -\frac{1}{18}\sin^3 a + \frac{43}{72}\sin^5 a$$

$$+ e^6 . -\frac{20}{81}\sin^3 a \cos a + \frac{1369}{405}\sin^5 a \cos a,$$

$$= e^5 . -\frac{1}{18}\left(1 + \frac{40}{9} e \cos a\right)\sin^3 a$$

$$+ e^5 . \frac{43}{72}\left(1 + \frac{10952}{1935} e \cos a\right)\sin^5 a ;$$

so that these terms may be represented by

$$-\frac{1}{18}\frac{e^5 \sin^3 a}{(1 - e\cos a)^{\frac{40}{9}}} + \frac{43}{72}\frac{e^5 \sin^5 a}{(1 - e\cos a)^{\frac{10952}{1935}}} . \quad . \quad . \quad (6)$$

If the exponent $\frac{10952}{1935}$ be replaced by $\frac{17}{3}$, the first two terms of the expanded value are

$$\frac{43}{72} e^5 \sin^5 a \left(1 + \frac{10965}{1935} e \cos a\right),$$

which differs from the quantity to be represented by only

$$\frac{13}{3240} e^6 \sin^5 a \cos a = \frac{1}{249} e^6 \sin^5 a \cos a, \text{ nearly.}$$

Thus the expression

$$a + \mathrm{A},$$

where

$$\mathrm{A} = \frac{e \sin a}{1 - e\cos a} - \frac{1}{2}\frac{e^3 \sin^3 a}{(1 - e\cos a)^{\frac{10}{3}}} - \frac{1}{18}\frac{e^5 \sin^3 a}{(1 - e\cos a)^{\frac{40}{9}}} + \frac{43}{72}\frac{e^5 \sin^5 a}{(1 - e\cos a)^{\frac{17}{3}}} \quad (7)$$

differs from the root of the equation (1) by

$$\frac{13}{3240} e^6 \sin^5 a \cos a + \text{terms of the seventh and higher orders,}$$

so that practically (7) gives the solution of Kepler's problem to the sixth order inclusive.

§ 4. Supposing the values of log e, log sin a, and log cos a to be given (in addition, of course, to the logarithms of the coefficients, which are assumed to be known), the calculation from (2) as it stands, up to the sixth order inclusive, would require twelve entries of the table of logarithms, while the calculation of log $(1 - e \cos a)$ requires two entries, and that of the terms in A four more, so that by the use of A six entries are saved. If log $(1 - e \cos a)$ be calculated and used in obtaining the first column of (2), the other terms in (2) require six entries, making nine in all, so that three are saved by the use of the last three terms of A. The calculation, of the expression (4), which

gives the value of the root up to the fourth order inclusive, requires four entries, while the calculation from (2) requires six entries if the values of the separate terms in the first column are obtained, and five entries if they are replaced by

$$\frac{e \sin a}{1 - e \cos a}.$$

§ 5. On developing the expression A up to the seventh order, it is found to be

$$\begin{aligned}
= \;& e\,.\sin a \\
&+ e^2.\sin a \cos a \\
&+ e^3.\sin a \cos^2 a - \frac{1}{2} \sin^3 a \\
&+ e^4.\sin a \cos^3 a - \frac{5}{3} \sin^3 a \cos a \\
&+ e^5.\sin a \cos^4 a - \frac{65}{18} \sin^3 a \cos^2 a - \frac{1}{18} \sin^3 a + \frac{43}{72} \sin^5 a \\
&+ e^6.\sin a \cos^5 a - \frac{520}{81} \sin^3 a \cos^3 a - \frac{20}{81} \sin^3 a \cos a + \frac{731}{216} \sin^5 a \cos a \\
&+ e^7.\sin a \cos^6 a - \frac{2470}{243} \sin^3 a \cos^4 a - \frac{490}{729} \sin^3 a \cos^2 a + \frac{3655}{324} \sin^5 a \cos^2 a \\
&+ \&c.
\end{aligned}$$

$$\begin{aligned}
= \;& e\,.\sin a \\
&+ e^2.\sin a \cos a \\
&+ e^3.\sin a \cos^2 a - \frac{1}{2} \sin^3 a \\
&+ e^4.\sin a \cos^3 a - \frac{5}{3} \sin^3 a \cos a \\
&+ e^5.\sin a \cos^4 a - \frac{11}{3} \sin^3 a \cos^2 a + \frac{13}{24} \sin^5 a \\
&+ e^6.\sin a \cos^5 a - \frac{20}{3} \sin^3 a \cos^3 a + \frac{2033}{648} \sin^5 a \cos a \\
&+ e^7.\sin a \cos^6 a - \frac{7900}{729} \sin^3 a \cos^4 a + \frac{30935}{2916} \sin^5 a \cos^2 a \\
&+ \&c.
\end{aligned}$$

Subtracting this from (2), it appears that the correction to be *added* to A to give the true value of the root of the equation (1) is

$$\begin{aligned}
& e^6. - \frac{13}{3240} \sin^5 a \cos a \\
&+ e^7. \quad \frac{5}{1458} \sin^3 a \cos^4 a + \frac{4363}{29160} \sin^5 a \cos^2 a - \frac{541}{720} \sin^7 a \\
&+ \&c.,
\end{aligned}$$

which very nearly

$$= e^6 . -\frac{1}{249} \sin^5 a \cos a$$

$$+ e^7 . \quad \frac{1}{292} \sin^3 a \cos^4 a + \frac{3}{20} \sin^5 a \cos^2 a - \frac{3}{4} \sin^7 a$$

$$+ \&c., \qquad \ldots \ldots \quad (8)$$

The smallness of the coefficients of the terms of the seventh order in (8) compared with those in (2) is worthy of remark: the sum of the first three coefficients in the e^7 line in (2) is equal to about 23 and in (8) to less than unity.

It is noticeable also how nearly the expression

$$-\frac{1}{2} \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}}$$

represents the first *three* terms of the second column of (2), the coefficient of the third term differing from $\frac{11}{3}$ by only $\frac{1}{18}$. The quantity to be added to the expression in (4) to give the value of the root up to the fifth order inclusive is

$$e^5 . -\frac{1}{18} \sin^3 a + \frac{43}{72} \sin^5 a,$$

which

$$= e^5 . -\frac{1}{18} \sin^3 a + \frac{3}{5} \sin^5 a, \text{ nearly,}$$

since

$$\frac{3}{5} - \frac{43}{72} = \frac{1}{360}.$$

§6. The numbers within square brackets denoting the Briggian logarithms of the coefficients expressed in seconds, the value of A is

$$[5{\cdot}3144251] \frac{e \sin a}{1 - e \cos a} - [5{\cdot}0133951] \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}} - [4{\cdot}0591526] \frac{e^5 \sin^3 a}{(1 - e \cos a)^{\frac{40}{9}}}$$

$$+ [5{\cdot}0905611] \frac{e^5 \sin^5 a}{(1 - e \cos a)^{\frac{17}{3}}}.$$

As an example of the application of this formula, let $a = \frac{1}{4}$ and $e = 50°$, this being the example considered on pp. 376, 377 of this volume.

The work is then as follows, every figure that has to be written down being printed:—

$\log e$	$= 9{\cdot}3979400$	$\log \cos a$	$= 9{\cdot}8080675$
„ e^3	$= 8{\cdot}1938200$	„ $\sin a$	$= 9{\cdot}8842540$
„ e^5	$= 6{\cdot}9897000$	„ $\sin^3 a$	$= 9{\cdot}6527620$
		„ $\sin^5 a$	$= 9{\cdot}4212700$

$\log e$	9·3979400	
„ $\cos a$	9·8080675	
	9·2060075	
$e \cos a$	0·1606969	
$1 - e \cos a$	0·8393031	
$\log (1 - e \cos a)$	9·9239189	
Ar. Co.	0·0760811	$= \log P$, say,
	0·008453456	$= \log P^{\frac{1}{9}}$
	0·25360368	$= \log P^{\frac{30}{9}} = \log P^{\frac{10}{3}}$
	0·33813824	$= \log P^{\frac{40}{9}}$
	0·42267280	$= \log P^{\frac{50}{9}}$
	0·43112626	$= \log P^{\frac{51}{9}} = \log P^{\frac{17}{3}}$

(+)	(−)	(−)	(+)
5·3144251	5·0133951	4·05915	5·09056
9·3979400	8·1938200	6·98970	6·98970
9·8842540	9·6527620	9·65276	9·42127
0·0760811	0·2536037	0·33814	0·43113
4·6727002	3·1135808	1·03975	1·93266

$''$	$''$	$''$	$''$
47065·23	1298·91	10·96	85·64
85·64	10·96		
47150·87	1309·87		
1309·87			
45841·00 $= 12^\circ\ 44'\ 1''{\cdot}00$;			

thus $A = 12^\circ\ 44'\ 1''{\cdot}00$, and

$$x = a + A, = 62^\circ\ 44'\ 1''{\cdot}00, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

the true value being (see p. 377) $62^\circ\ 43'\ 56''{\cdot}00$.

The value of the expression in (4), that is of a + the first two terms of $A, = 62^\circ\ 42'\ 46''$, and the correction for terms of the fifth order, viz.:

$$e^5 . -\frac{1}{18} \sin^3 a + \frac{3}{5} \sin^5 a$$

$= -5''{\cdot}03 + 31''{\cdot}88$, giving $x = 62^\circ\ 43'\ 13''$.

§7. The correction to be applied to (7) in order to include approximately the terms of the seventh order, viz.:

$$e^7 . \frac{3}{20} \sin^5 a - \frac{9}{10} \sin^7 a$$

$= 0''\cdot50 - 1''\cdot75 = -1''\cdot25$. As the error of (9) is about $5''$, it will be seen that, the seventh order correction just obtained being very small, there is very little advantage in including these terms. In general, the classification of the terms according to orders of e supposes that e is very small; so small, in fact, that a term involving e^{n+1} may be regarded as distinctly smaller than one involving e^n: but if e be as large as $\frac{1}{4}$, the terms of the orders near to e^8 do not differ much from one another in magnitude, in consequence of the increased values of the coefficients, which somewhat counteract the effect of the additional factor e. This is readily seen by considering the terms in e^8. By expanding the terms in A, it will be found that the terms in e^8, omitting the first, are

$$= -14\cdot90809 \sin^3 a \cos^5 a - 1\cdot44389 \sin^3 a \cos^3 a + 28\cdot82887 \sin^5 a \cos^3 a$$

$$= -16\cdot35198 \sin^3 a \cos^5 a + 27\cdot38498 \sin^5 a \cos^3 a \quad . \; . \; . \; . \; . \; . \; . \; (10)$$

(in which the coefficients are expressed as decimals so that their magnitudes may be more evident): the terms in e^8 in (2), omitting the first, are

$$= -16\cdot33333 \sin^3 a \cos^5 a + 28\cdot46667 \sin^5 a \cos^3 a - 6\cdot21270 \sin^7 a \cos a \quad . \; . \; . \; . \; (11)$$

and the difference

$$= e^8 . \, 0\cdot01865 \sin^3 a \cos^5 a + 1\cdot08169 \sin^5 a \cos^3 a - 6\cdot21270 \sin^7 a \cos a:$$

in the case of $e = \frac{1}{4}$, this difference $= 0''\cdot00 + 0''\cdot24 - 1''\cdot95 = -1''\cdot71$.

The close agreement between the coefficients of $\sin^3 a \cos^5 a$ and $\sin^5 a \cos^3 a$ in (10) and (11) is remarkable; and it will be noted that the coefficient of the new term, viz. the term in $\sin^7 a \cos a$, is, as in the case of the terms of the seventh order, much the largest.

It thus appears that if e be so large that, in order to obtain the value of x to the nearest tenth or hundredth of a second, terms of a higher order than the sixth have to be included, in general no great additional accuracy will be obtained by applying to A the corrections for the terms of the seventh and eighth orders, while of course if the terms in e^7 may be neglected, the corrections are insensible. Whenever, therefore, the series in (2) affords a suitable solution of the problem, the expression $a + A$ gives the solution up to the seventh order, and in general but little additional accuracy is gained by applying the corrections

for the seventh and higher orders. The advantage of using the formula $a+A$ would be greatest if e were very small, so that the order of e governed the magnitude of each term, and if the value of x were required with great exactness; as, for example, if the accuracy required were such that ten-figure logarithms had to be employed in the calculation of the first term of A, but that terms above e^7 were insensible.

The following are the values of the terms in (2), for this case of $e=\frac{1}{4}$, $a=50°$, up to the sixth order inclusive:—

$$\begin{aligned} x &= 50° \\ &\quad + 39502''\cdot00 \\ &\quad + \ 6347''\cdot85 \\ &\quad + \ 1020''\cdot08 - 724''\cdot40 \\ &\quad + \ \ 163''\cdot92 - 388''\cdot03 \\ &\quad + \ \ \ 26''\cdot34 - 137''\cdot18 + 28''\cdot78 \\ &\quad + \ \ \ \ 4''\cdot23 - \ 40''\cdot08 + 26''\cdot76 \\ &= 50° + 45830''\cdot27 = 62°\ 43'\ 50''\cdot27; \end{aligned}$$

the terms in e^7

$$\begin{aligned} &= +0''\cdot68 - 10''\cdot47 + 14''\cdot76 - 1''\cdot46 \\ &= 3''\cdot51; \end{aligned}$$

and the terms in e^8

$$\begin{aligned} &= 0''\cdot11 - \ 2''\cdot54 + \ 6''\cdot28 - 1''\cdot95 \\ &= 1''\cdot90; \end{aligned}$$

so that, when the seventh order terms are included, the value is 62° 43′ 53″·78; and when the terms of the eighth order are also included, 62° 43′ 55″·68.

§ 8. As another example, let $e=\frac{1}{10}$, $a=50°$, and the terms of A

$$= 16886''\cdot23 - 57''\cdot85 - 0''\cdot07 + 0''\cdot47,$$

so that $a+A = 54°\ 40'\ 28''\cdot78$, which will be found to satisfy the equation to within the hundredth of a second.

§ 9. The exponents in the denominators of the terms in A are $\frac{30}{9}, \frac{40}{9}, \frac{51}{9}$; if the last exponent be taken to be $\frac{50}{9}$, the sixth order error

$$= +\frac{101}{1620}e^6 \sin^5 a \cos a, \quad = +\frac{1}{16}e^6 \sin^5 a \cos a, \text{ nearly},$$

and is thus much larger than when the exponent is $\frac{51}{9}$, the error in this case (*see* § 3) being

$$= -\frac{1}{249}e^6 \sin^5 a \cos a, \text{ nearly}.$$

§ 10. If the terms in (2) be treated in columns as in §§ 2 and 3, each column being kept distinct, it will be found that up to the sixth order inclusive,

$$x = a + \frac{e \sin a}{1 - e \cos a} - \frac{1}{2} \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}} - \frac{1}{18} \frac{e^5 \sin^3 a \cos^2 a}{(1 - e \cos a)^{\frac{40}{9}}} + \frac{13}{24} \frac{e^5 \sin^5 a}{(1 - e \cos a)^{\frac{376}{65}}},$$

so that

$$x = a + \mathrm{B},$$

where

$$\mathrm{B} = \frac{e \sin a}{1 - e \cos a} - \frac{1}{2} \frac{e^3 \sin^3 a}{(1 - e \cos a)^{\frac{10}{3}}} - \frac{1}{18} \frac{e^5 \sin^3 a \cos^2 a}{(1 - e \cos a)^{\frac{40}{9}}} + \frac{13}{24} \frac{e^5 \sin^5 a}{(1 - e \cos a)^{\frac{23}{4}}},$$

the correction to be added to B being

$$\begin{aligned} & e^6 . \frac{9}{480} \sin^5 a \cos a \\ & + e^7 . \frac{5}{1458} \sin^3 a \cos^4 a + \frac{947}{3840} \sin^5 a \cos^2 a - \frac{541}{720} \sin^7 a \\ & + \&c., \end{aligned}$$

which, very nearly,

$$\begin{aligned} = \quad & e^6 . \frac{1}{53} \sin^5 a \cos a \\ & + e^7 . \frac{1}{292} \sin^3 a \cos^4 a + \frac{1}{4} \sin^5 a - \sin^7 a \\ & + \&c. \end{aligned}$$

It is clear that the expression B is not so convenient for calculation as A; and also the residual sixth order term is larger for B than for A.

§ 11. In t. v. (1859) of the *Annales de l'Observatoire de Paris*, pp. 349, 350,* M. Serret gives an account of a method, indicated by M. Leverrier on p. 192 of t. i., of solving Kepler's problem by means of a series of a different form to (2). The method is as follows:—

Let x_0 be an approximate value of x, and put

$$v = a - x_0 + e \sin x_0;$$

then

$$x - e \sin x = x_0 - e \sin x_0 + v.$$

This is the equation to be solved, and we can calculate x by

* "Note sur l'équation dont dépend l'anomalie excentrique, et sur les séries qui se présentent dans la théorie du mouvement elliptique des corps célestes."

developing it in ascending powers of v by Maclaurin's theorem; we have

$$(1-e\cos x)\frac{dx}{dv}=1,$$

$$(1-e\cos x)\frac{d^2x}{dv^2}+\ e\sin x\left(\frac{dx}{dv}\right)^2=0,$$

$$(1-e\cos x)\frac{d^3x}{dv^3}+3e\sin x\frac{dx}{dv}\frac{d^2x}{dv^2}+e\cos x\left(\frac{dx}{dv}\right)^3=0,$$

&c. &c.

When $v=0$, $x=x_0$, and

$$\frac{dx}{dv}=\frac{1}{1-e\cos x_0},\quad \frac{d^2x}{dv^2}=-\frac{e\sin x_0}{(1-e\cos x_0)^3},$$

$$\frac{d^3x}{dv^3}=\frac{3e^2\sin^2 x_0}{(1-e\cos x_0)^5}-\frac{e\cos x_0}{(1-e\cos x_0)^4},$$

whence

$$x=x_0+\frac{1}{1-e\cos x_0}v-\frac{1}{2}\frac{e\sin x_0}{(1-e\cos x_0)^3}v^2$$
$$+\left\{\frac{1}{2}\frac{e^2\sin^2 x_0}{(1-e\cos x_0)^5}-\frac{1}{6}\frac{e\cos x_0}{(1-e\cos x_0)^4}\right\}v^3+\text{\&c.}, \quad \ldots\ldots \quad (12)$$

In the case of an orbit in which e is very small, put $x_0=a$, then $v=e\sin a$ and, neglecting terms of the fourth order,

$$x=a+\frac{e\sin a}{1-e\cos a}-\frac{1}{2}\left(\frac{e\sin a}{1-e\cos a}\right)^3. \quad \ldots\ldots \quad (13)$$

M. Serret remarks that this formula is very convenient for calculation, and that the last term can be immediately derived from the previous term.

§ 12. It will be found that the coefficient of v^4 in (12) is

$$-\frac{5}{8}\frac{e^3\sin^3 x_0}{(1-e\cos x_0)^7}+\frac{5}{12}\frac{e^2\sin x_0\cos x_0}{(1-e\cos x_0)^6}+\frac{e\sin x_0}{(1-e\cos x_0)^5};$$

and therefore, putting $x_0=a$, the formula, whose first three terms are given by M. Serret's equation (13), is

$$x=a+\frac{e\sin a}{1-e\cos a}-\frac{1}{2}\frac{e^3\sin^3 a}{(1-e\cos a)^3}$$
$$+\frac{1}{2}\frac{e^5\sin^5 a}{(1-e\cos a)^5}-\frac{1}{6}\frac{e^4\sin^3 a\cos a}{(1-e\cos a)^4}$$
$$-\frac{5}{8}\frac{e^7\sin^7 a}{(1-e\cos a)^7}+\frac{5}{12}\frac{e^6\sin^5 a\cos a}{(1-e\cos a)^6}+\frac{1}{24}\frac{e^5\sin^5 a}{(1-e\cos a)^5}$$
$$+\text{\&c.}; \quad \ldots\ldots \quad (14$$

the second line being derived from the terms in v^2, and the third from those in v^3.

In the case of $e = \frac{1}{4}$, $a = 50°$, this gives

$$
\begin{aligned}
x = 50° &+ 47065''\cdot23 - 1225''\cdot24 \\
&+ \quad 63''\cdot79 - \quad 78''\cdot20 \\
&- \quad\ 4''\cdot15 + \quad 9''\cdot16 + 5''\cdot32 \\
&+ \&c.
\end{aligned}
$$

		°	′	″
The first line	=	52	43	59·99
the second line	= −			14·41
the third line	= +			10·33
		52	43	55·91

The close approximation afforded by the first line (the error is only $3''\cdot99$) is very remarkable, but it is of course accidental; the term of the fourth order $= 78''\cdot20$, and those of the fifth order amount to $69''\cdot11$.

§ 13. The formula (13), which is true to the third order, only differs from (4), which is true to the fourth order, in the exponent of the denominator, which is $\frac{10}{3}$ instead of 3. If $e = \frac{1}{10}$, $a = 50°$, the last term of (13) $= 56''\cdot59$; and the value of x from (13) is $54° 40' 29''\cdot64$, and from (4) is $54° 40' 28''\cdot38$, the true value as found in § 8 being $54° 40' 28''\cdot78$.

As another example, let $e = \frac{1}{7}$, a being, as before, $50°$, then (14) gives

$$
\begin{aligned}
x = 50° &+ 24854''\cdot92 - 180''\cdot45 \\
&+ \quad 2''\cdot62 - \quad 6''\cdot08 \\
&- \quad 0''\cdot05 + \quad 0''\cdot22 + 0''\cdot22 \\
&+ \&c.;
\end{aligned}
$$

the first line $= 56° 51' 14''\cdot47$, the second $- 3''\cdot46$, and the third $0''\cdot39$, leading to $x = 56° 51' 11''\cdot40$; and the true value obtained by substituting this in (1) is $56° 51' 11''\cdot39$.

The value of $a + A$ is

$$50° + 24854''\cdot92 - 186''\cdot34 - 0''\cdot47 + 3''\cdot34,$$

which $= 56° 51' 11''\cdot45$.

The value obtained from (4) is $56° 51' 8''\cdot58$.

§ 14. It will be noticed that in both these examples the formula (13) affords a very close approximation to the true value; this is due to the fact that the terms in the second line of (14) nearly counteract one another for the values of e and a that have been taken; for, these terms

$$= \frac{1}{2} \frac{e^4 \sin^3 a}{(1 - e \cos a)^4} \{e \sin^2 a - \frac{1}{3} \cos a\},$$

and the term in brackets is small for the assumed values of e and a. As an example in which this is not the case, let $e = \frac{1}{4}$, $a = 130°$, then (14) gives

$$\begin{aligned} &130° + 34032''\cdot 99 - 463''\cdot 25 \\ &\quad + \quad 12''\cdot 61 + 21''\cdot 38 \\ &\quad - \quad 0''\cdot 43 - 1''\cdot 46 + 1''\cdot 05. \end{aligned}$$

The first line $= 139° \; 19' \; 29''\cdot 74$, the second $33''\cdot 99$, and the third $-0''\cdot 84$, leading to $x = 139° \; 20' \; 2''\cdot 89$.

The value of $a + \text{A}$

$$\begin{aligned} &= 130° + 34032''\cdot 99 - 440''\cdot 81 - 2''\cdot 59 + 13''\cdot 64 \\ &= 139° \; 20' \; 3''\cdot 23. \end{aligned}$$

The value of x given by (4) is $139° \; 19' \; 52''\cdot 18$, which only differs from the true result (viz. $139° \; 20' \; 2''\cdot 93$) by about $11''$. It is evident that the effect of the denominators in (7) and (14) is to increase the magnitudes of the terms if a be in the first or fourth quadrants, and diminish them if a be in the second or third quadrants.

The expression in (4) is only very slightly more complicated than (13), and as it represents the root to one order higher, it is in general to be preferred.

§ 15. The series in (14) is clearly the expansion of x in powers of $\frac{e}{1 - e \cos a}$, and it may therefore be obtained directly in the following manner:—

Let

$$\epsilon = \frac{e}{1 - e \cos a},$$

then

$$e = \frac{\epsilon}{1 + \epsilon \cos a};$$

the equation is

$$x = a + \frac{\epsilon}{1 + \epsilon \cos a} \sin x,$$

viz.

$$x = a + \epsilon (\sin x - x \cos a + a \cos a),$$

and therefore, by Lagrange's theorem,

$$\begin{aligned} x = a + \epsilon \sin a + \frac{\epsilon^2}{1.2} \frac{d}{dx} (\sin x - x \cos a + a \cos a)^2 \\ + \frac{\epsilon^3}{1.2.3} \frac{d^2}{dx^2} (\sin x - x \cos a + a \cos a)^3 + \&c., \quad . \quad . \quad . \quad (15) \end{aligned}$$

x being put equal to a after the differentiations have been performed; thus

$$\begin{aligned} x = {} & a \\ & + \epsilon \,. \; \sin a \\ & + \epsilon^3 . \; -\frac{1}{2} \sin^3 a \\ & + \epsilon^4 . \; -\frac{1}{6} \sin^3 a \cos a \\ & + \&c. \end{aligned}$$

It is evident that there will be no terms in the expansion involving powers of $a \cos a$, for

$$\sin x - x \cos a + a \cos a = \sin a, \text{ when } x = a,$$

and its differential coefficient $= \cos x - \cos a$. When $x = a$ this vanishes, and the terms are in consequence greatly simplified: in fact, the development of (15) may be very rapidly effected.

§ 16. A similar method affords the expansion of x in powers of $\dfrac{e}{1 + e \cos a}$; for, let

$$\eta = \frac{e}{1 + e \cos a},$$

then

$$x = a + \frac{\eta}{1 - \eta \cos a} \sin x,$$

viz.

$$x = a + \eta (\sin x + x \cos a - a \cos a),$$

and therefore

$$\begin{aligned} x = a + \eta \sin a + {} & \frac{\eta^2}{1.2} \frac{d}{dx} (\sin x + x \cos a - a \cos a)^2 \\ & + \frac{\eta^3}{1.2.3} \frac{d^2}{dx^2} (\sin x + x \cos a - a \cos a)^3 + \&c., \end{aligned}$$

x being put equal to a.

We thus find

$$\begin{aligned} x = {} & a \\ & + \eta \,. \; \sin a \\ & + \eta^2 . \; 2 \sin a \cos a \\ & + \eta^3 . \; 4 \sin a \cos^2 a - \frac{1}{2} \sin^3 a \\ & + \eta^4 . \; 8 \sin a \cos^3 a - \frac{19}{6} \sin^3 a \cos a \\ & + \eta^5 . \; 16 \sin a \cos^4 a - \frac{40}{3} \sin^3 a \cos^2 a + \frac{13}{24} \sin^5 a \\ & + \&c. \end{aligned}$$

The first column evidently

$$= a + \frac{\eta \sin a}{1 - 2\eta \cos a}, \quad = a + \frac{e \sin a}{1 - e \cos a}.$$

§ 17. I here add another solution of Kepler's problem in series, the denominators of the terms being powers of $1 - e \cos a$.

If

$$x = a + efx$$

then, by Lagrange's theorem,

$$\mathrm{F}x = \mathrm{F}a + efa\mathrm{F}'a + \frac{e^2}{1.2}\frac{d}{da}f^2a\mathrm{F}'a + \frac{e^3}{1.2.3}\frac{d^2}{da^2}f^3a\mathrm{F}'a + \&c.$$

Now let

$$\mathrm{F}x = \frac{1}{1 - ef'x} = \frac{dx}{da},$$

therefore

$$\frac{dx}{da} = \frac{1}{1 - ef'a} + \frac{e^2 faf''a}{(1 - ef'a)^2} + \frac{e^3}{1.2}\frac{d}{da}\frac{f^2af''a}{(1 - ef'a)^2} + \frac{e^4}{1.2.3}\frac{d^2}{da^2}\frac{f^3af''a}{(1 - ef'a)^2} + \&c.$$

$$= 1 + \frac{ef'a}{1 - ef'a} + \frac{e^2 faf''a}{(1 - ef'a)^2} + \frac{e^3}{1.2}\frac{d}{da}\frac{f^2af''a}{(1 - ef'a)^2} + \&c.;$$

whence, integrating,

$$x = a + \frac{efa}{1 - ef'a} + \frac{e^3}{1.2}\frac{f^2af''a}{(1 - ef'a)^2} + \frac{e^4}{1.2.3}\frac{d}{da}\frac{f^3af''a}{(1 - ef'a)^2} + \&c.$$

In the case of Kepler's problem

$$fa = \sin a, \quad f'a = \cos a, \quad f''a = -\sin a;$$

and therefore

$$x = a + \frac{e \sin a}{1 - e \cos a} - \frac{e^3}{1.2}\frac{\sin^3 a}{(1 - e \cos a)^2} - \frac{e^4}{1.2.3}\frac{d}{da}\frac{\sin^4 a}{(1 - e \cos a)^2}$$

$$- \frac{e^5}{1.2.3.4}\frac{d^2}{da^2}\frac{\sin^5 a}{(1 - e \cos a)^2} - \&c.$$

$$= a + \frac{e \sin a}{1 - e \cos a}$$

$$- \frac{e^3}{2}\frac{\sin^3 a}{(1 - e \cos a)^2}$$

$$- \frac{e^4}{3}\left\{2\frac{\sin^3 a \cos a}{(1 - e \cos a)^2} - \frac{e \sin^5 a}{(1 - e \cos a)^3}\right\},$$

$$- \frac{e^5}{24}\left\{20\frac{\sin^3 a}{(1 - e \cos a)^2} - 25\frac{\sin^5 a}{(1 - e \cos a)^2} - 22\frac{e \sin^5 a \cos a}{(1 - e \cos a)^3} + 6\frac{e^2 \sin^7 a}{(1 - e \cos a)^4}\right\},$$

$$- \&c.$$

1877, *August* 4.